

Fluctuating topological invariants in string theory as an Abelian gauge theory

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It is shown that the topological invariants associated with the two-dimensional world-surface in string theory have nontrivial fluctuations around their nonexistent classical dynamics. Additionally it is proved that the underlying geometrical structure in a covariant phase space formulation for such topological string actions mimics entirely that of an Abelian gauge theory.

I. Introduction

Recently [1, 2] it has been demonstrated that the presence of topological terms in Lagrangians for string theory has a dramatic effect on the covariant phase space formulation of the theory, despite such terms do not have an effective contribution on the equations of motion. This fact is not certainly exclusive of string theory; for example, in [3] the relevant role that the topological terms play in the covariant canonical formalism for 4-dimensional BF theory has been explored, and additionally it is clarified that the knowledge of the equations of motion for a physical system is not enough for specifying the symplectic properties of the phase space, but an action principle is a necessary ingredient for such a purpose. An immediate consequence of these results is that the presence of topological terms will lead in general to a completely different quantum field theory.

Traditionally the topological terms have been considered as corrective or additional terms to other Lagrangian terms, which normally have nontrivial equations of motion. Subsequently one will try, for example, to observe the shift on the resultant quantum field theory. In the present work we attempt to explore an extreme situation, which was suggested in [2], and consists in considering that the only Lagrangian term in a string action is a topological invariant associated with the 2-

dimensional manifold of the worldsheet. We want to show then that such a topological string action has, by itself, a nontrivial covariant phase space formulation, and consequently a nontrivial quantum field theory. The specific result will be that the underlying symplectic structure of the topological action mimics entirely that of an Abelian gauge theory, with all what such a result might imply.

The philosophy behind this work is that the classical equations of motion are not the most important thing in physics; in fact, as we shall see, we do not need, at all, such equations *for making physics*. In some sense, we are not making something new, since it is known that in a cohomological topological field theory of the Witten type one tries to make physics from topological invariants of certain manifolds, which have trivial classical dynamics, but the physics is found in other domain (see for example [4]).

This work is organized as follows. In the next section we give an outline of the differential geometry of an embedding developed by Carter [5], on which we base our calculations. In Section III we summarize briefly the relevant results of [2] that are important in the present context. In Section IV we show the existence of a nontrivial fluctuation dynamics for a topological invariant, and a covariantly conserved current is constructed from it, which will be identified with the integral kernel of the symplectic structure for the topological string action. In Section V we prove that the integral kernel is obtained also from the variations of a symplectic potential, following the ideas discussed previously in [2]. In Section VI we describe the analogy between the symplectic structure of the topological invariant and that of an Abelian gauge theory. In Section VII the analogy is extended to the symmetry properties of the geometrical structures. In Section VIII we discuss how to extend the definition of covariant phase space for topological terms without classical equations of motion. We finish in Section IX with some remarks and prospects.

II. Basic differential geometry of an imbedding

In this Section, we outline the description given in [5] for the intrinsic curvature that is associated with a spacelike or timelike p -surface imbedded in an n -dimensional space or spacetime background with metric $g_{\mu\nu}$. Specifically the *internal curvature tensor* of the imbedding can be written as

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2n_{\sigma}{}^{\mu} n_{\nu}{}^{\tau} n_{[\lambda}{}^{\pi} \overline{\nabla}_{\kappa]} \rho_{\pi}{}^{\sigma}{}_{\tau} + 2\rho_{[\kappa}{}^{\mu\pi} \rho_{\lambda]\pi\nu}, \quad (1)$$

where $n^{\mu\nu}$ is the (first) *fundamental tensor* of the p -surface, that together with the complementary orthogonal projection $\perp^{\mu\nu}$ satisfy

$$n^\mu{}_\nu + \perp^\mu{}_\nu = g^\mu{}_\nu, \quad n^\mu{}_\nu \perp^\nu{}_\rho = 0, \quad (2)$$

and the tangential covariant differentiation operator is defined in terms of the fundamental tensor as

$$\bar{\nabla}_\mu = n^\rho{}_\mu \nabla_\rho, \quad (3)$$

where ∇_ρ is the usual Riemannian covariant differentiation operator associated with $g_{\mu\nu}$. Additionally, $\rho_\lambda{}^\mu{}_\nu$ represents the background spacetime components of the internal frame components of the natural gauge connection for the group of p -dimensional internal frame rotations. It satisfies the properties

$$\rho_{\lambda\mu\nu} = -\rho_{\lambda\nu\mu}, \quad \perp^\rho{}_\lambda \rho_{\rho\mu\nu} = 0 = \perp^\rho{}_\lambda \rho_{\mu\rho\nu}, \quad (4)$$

whereas the internal curvature tensor (1) satisfies the usual Riemann symmetry properties and the Ricci contractions

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma, \quad R = R_\sigma{}^\sigma, \quad (5)$$

with

$$\perp^\sigma{}_\beta R_{\sigma\lambda\mu\nu} = 0, \quad \perp^\sigma{}_\beta R_{\sigma\mu} = 0. \quad (6)$$

From the fundamental tensor and the Ricci contractions (5) one can define the internal adjusted Ricci tensor as

$$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2(p-1)} R n_{\mu\nu}, \quad (7)$$

where p is the dimension of the imbedded p -surface. As pointed out in [5], for the special case $p = 2$ of a two-dimensional imbedded surface (that applies to string theory, for which this work is concerned), the adjusted Ricci tensor (7) vanishes identically:

$$\tilde{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R n_{\mu\nu} = 0. \quad (8)$$

Eq. (8) will imply, as we shall see below, that the inner curvature scalar given in (5) can not give any effective contribution in a variational principle.

Additionally, the second fundamental tensor is defined by [5]

$$K_{\lambda\mu}{}^\nu = n_\mu^\sigma \bar{\nabla}_\lambda n_\sigma^\nu = K_{(\lambda\mu)}{}^\nu, \quad (9)$$

with its property of tangentiality of the first two indices and orthogonality of the last, and one defines then the curvature vector as

$$K^\rho = K_\nu{}^{\nu\rho}, \quad n_\rho^\mu K^\rho = 0, \quad (10)$$

and

$$K_\nu{}^{\rho\nu} = 0. \quad (11)$$

The third fundamental tensor

$$\Theta_{\kappa\lambda\mu}{}^\nu = n_\lambda^\rho n_\mu^\sigma \perp_\tau^\nu \bar{\nabla}_\kappa K_{\rho\sigma}{}^\tau = \Theta_{\kappa(\lambda\mu)}{}^\nu, \quad (12)$$

satisfies

$$\begin{aligned} \bar{\nabla}_\kappa K_{\lambda\mu}{}^\nu &= \Theta_{\kappa\lambda\mu}{}^\nu + 2K_\kappa{}^\sigma{}_{(\lambda} K_{\mu)\sigma}{}^\nu - K_\kappa{}^\nu{}_\sigma K_{\lambda\mu}{}^\sigma, \\ 2\Theta_{[\kappa\lambda]\mu}{}^\nu &= n_\kappa^\rho n_\lambda^\sigma n_\mu^\tau \perp_\gamma{}^\nu B_{\rho\sigma}{}^\gamma{}_\tau, \end{aligned} \quad (13)$$

where $B_{\rho\sigma}{}^\gamma{}_\tau$ is the background Riemann curvature tensor.

Finally, we need to obtain from the Bianchi identity for the internal curvature (in a general background) [5],

$$n_{[\kappa}{}^\nu n_\lambda^\rho \bar{\nabla}_{\mu]} R_{\nu\rho}{}^{\sigma\tau} = 2R_{[\kappa\lambda}{}^{\nu[\tau} K_{\mu]\nu}{}^{\sigma]},$$

its contracted version

$$\bar{\nabla}_\mu (2R^{\mu\nu} - Rn^{\mu\nu}) = (2R^{\sigma\rho} - Rn^{\sigma\rho}) K_{\sigma\rho}{}^\nu, \quad (14)$$

where the tangentiality and orthogonality of $K_{\lambda\mu}{}^\nu$ and the relation $\bar{\nabla}_\lambda n_{\mu\nu} = 2K_{\lambda(\mu\nu)}$ have been used.

For more details about this section, see directly Refs. [5, 6].

III. Preliminaries of the deformation dynamics

Within the covariant scheme given by Carter [6] for the fluctuation dynamics, it is known that

$$\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma} n^{\mu\nu} \delta g_{\mu\nu}, \quad (15)$$

where γ is the determinant of the embedded surface metric, and the variation of the background metric is given by its Lie derivative with respect to the deformation vector field $\xi^\mu = \delta X^\mu$ of the embedding,

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \quad (16)$$

Furthermore, in [2] it is shown, taking into account the gauge nature of ρ , that

$$\begin{aligned} \delta R_{\kappa\lambda}{}^\mu{}_\nu &= 2n_\sigma{}^\mu n_\nu{}^\tau n_{[\lambda}{}^\pi \bar{\nabla}_{\kappa]} \delta \rho_\pi{}^\sigma{}_\tau, \\ \delta R_{\mu\nu} &= 2n_\sigma{}^\kappa n_\nu{}^\tau n_{[\mu}{}^\pi \bar{\nabla}_{\kappa]} \delta \rho_\pi{}^\sigma{}_\tau, \\ n^{\mu\nu} \delta R_{\mu\nu} &= \bar{\nabla}_\mu \psi_{top}^\mu, \end{aligned} \quad (17)$$

where

$$\psi_{top}^\mu = n^{\alpha\beta} \delta \rho_{\alpha\beta}^\mu - n_\beta^\alpha n^{\mu\tau} \delta \rho_\alpha{}^\beta{}_\tau. \quad (18)$$

Using Eqs. (17), and (18), and considering that $R = n^{\mu\nu} R_{\mu\nu}$, in [2] it is shown that the variation of a Lagrangian term proportional to the inner curvature scalar of the imbedded p -surface

$$\chi = \sigma_1 \int \sqrt{-\gamma} R d\bar{\Sigma}, \quad (19)$$

is given by

$$\delta \chi = \sigma_1 \int \sqrt{-\gamma} \left(\frac{1}{2} R n^{\mu\nu} - R^{\mu\nu} \right) \delta g_{\mu\nu} d\bar{\Sigma} + \sigma_1 \int \bar{\nabla}_\mu \psi_{top}^\mu d\bar{\Sigma}; \quad (20)$$

where σ_1 is a fixed parameter.

In general, $\frac{1}{2} R n^{\mu\nu} - R^{\mu\nu}$ does not vanish for a geometry of arbitrary dimension, however, from Eq. (8), the adjusted Ricci tensor vanishes identically for string theory, and χ does not give dynamics to such objects. Hence χ is a topological invariant for string theory (for appropriate boundary conditions), that geometrically corresponds to the number of handles of the world-surface, the so called Euler characteristic.

Considering the orthogonal gauge $n_\nu^\mu \xi^\nu = 0$, which removes the nonphysically observable tangential of the deformation [6] (and that we consider throughout this work), the first term in Eq. (20)

can be rewritten (using Eq. (16)) as

$$\begin{aligned} \left(\frac{1}{2}R n^{\mu\nu} - R^{\mu\nu}\right)\delta g_{\mu\nu} &= \xi^\nu \bar{\nabla}_\mu (2R^\mu{}_\nu - R n^\mu{}_\nu) \\ &= (2R^{\sigma\rho} - R n^{\sigma\rho})K_{\sigma\rho\nu}\xi^\nu, \end{aligned} \quad (21)$$

where the last equality follows from the contracted Bianchi identity (14). Thus, Eq. (21) implies from Eq. (20) that the equation of the motion for χ is given by

$$\sigma_1(2R^{\sigma\rho} - R n^{\sigma\rho})K_{\sigma\rho\nu} = 0. \quad (22)$$

IV. Fluctuations of a topological invariant in an arbitrary background

The idea now is to calculate the deformations of the equations of motion (22) for an arbitrary p -brane and to impose on the deformation equations thus obtained the condition (8) that identifies the two-dimensional world-surface. The result will be that the equations governing the fluctuations of the topological invariant admit solutions different to the trivial ones, in spite of the null dynamics at the level of the unperturbed equations of motion.

Hence, the variations of the equations (22) are given by

$$\sigma_1 K_{\sigma\rho\nu} \delta(2R^{\sigma\rho} - R n^{\sigma\rho}) + \sigma_1 (2R^{\sigma\rho} - R n^{\sigma\rho}) \delta K_{\sigma\rho\nu} = 0, \quad (23)$$

where the second term will be not worked out, because will vanish under the condition (8). On the other hand, the variation of the adjusted Ricci tensor in the first term in Eq. (23) can be written in terms of the expressions (16), and (17) as

$$\begin{aligned} \delta(2R^{\sigma\rho} - R n^{\sigma\rho}) &= 2\bar{\mathcal{C}}^{\mu\nu\sigma\rho}\delta R_{\mu\nu} + (R^{\alpha\beta}n^{\sigma\rho} - 2n^{\rho\alpha}R^{\sigma\beta})\delta g_{\alpha\beta} \\ &\quad - (2R^{\beta\rho} - R n^{\beta\rho})n^{\sigma\alpha}\delta g_{\alpha\beta}, \end{aligned} \quad (24)$$

where the third term vanishes under the condition (8), and the other ones are reduced to

$$\delta(2R^{\sigma\rho} - R n^{\sigma\rho}) = \bar{\mathcal{C}}^{\mu\nu\sigma\rho}(2\delta R_{\mu\nu} - R \delta g_{\mu\nu}), \quad (25)$$

under the same condition (8). $\overline{\mathcal{C}}^{\mu\nu\sigma\rho}$ is the called hyper-Cauchy tensor introduced by Carter [7]:

$$\begin{aligned}\overline{\mathcal{C}}^{\mu\nu\sigma\rho} &\equiv n^{\sigma(\mu} n^{\nu)\rho} - \frac{1}{2} n^{\mu\nu} n^{\sigma\rho} \\ &= \overline{\mathcal{C}}^{(\mu\nu)(\sigma\rho)}.\end{aligned}\tag{26}$$

From Eq. (25) we immediately conclude that, although the adjusted Ricci tensor does vanish for string theory, its fluctuations do not necessarily. In fact, the existence of nontrivial fluctuations for a topological term is a generic issue in all physical systems [2], and one may make the corresponding for any system of interest.

For our proposes, we need to work out more Eqs. (23), in order to express them explicitly in terms of the embedding deformation vector ξ^μ , and thus it is essential to know the deformations of the frame vectors $\{i_A^\mu\}$ tangential to the world-surface:

$$\delta i_A^\mu = i_A^\alpha K_{\alpha}{}^\mu{}_\lambda \xi^\lambda,\tag{27}$$

which implies that the deformations of the internal connection $\rho_\lambda{}^\mu{}_\nu$ is given, after a long calculation, by

$$\begin{aligned}\delta \rho_\lambda{}^\mu{}_\nu &= K_{\lambda\nu\alpha} \overline{\nabla}^\mu \xi^\alpha - K_{\lambda}{}^\mu{}_\alpha \overline{\nabla}_\nu \xi^\alpha - n_\rho^\mu n_\nu^\sigma n_\lambda^\tau \mathcal{B}^\rho{}_{\sigma\tau\alpha} \xi^\alpha \\ &= \overline{\nabla}^\mu (K_{\nu\lambda\alpha} \xi^\alpha) - \overline{\nabla}_\nu (K_{\lambda\alpha}{}^\mu \xi^\alpha) + 2K_{\nu}{}^\rho{}_{(\lambda} K^{\mu)}{}_{\rho\alpha} \xi^\alpha - 2K^{\mu\rho}{}_{(\lambda} K_{\nu)\rho\alpha} \xi^\alpha.\end{aligned}\tag{28}$$

Note that Eq. (28) is a manifestly tensorial expression for the deformation of the connection ρ , with support confined to the worldsheet; $\delta\rho$ also turns out to be antisymmetric on the indices μ, ν , such as $\rho_\lambda{}^\mu{}_\nu$ itself. The last equality in Eq. (28) is obtained using the relation (13), and it shows in a manifest form that the deformation $\delta\rho$ is expressed in terms of the fundamental field of the deformation dynamics, $K_\alpha{}^\mu{}_\lambda \xi^\lambda$, which is clear also in Eq. (27), and will be a rule throughout the present treatment.

Therefore, from Eqs. (17), and (28), we can determine the (tangential projection) of the deformation of the internal Ricci tensor,

$$\begin{aligned}n_\alpha^\mu n_\beta^\nu \delta R_{\mu\nu} &= n_\alpha^\mu n_\beta^\nu [\overline{\nabla}_\sigma \overline{\nabla}^\sigma (K_{\mu\nu\lambda} \xi^\lambda) - \overline{\nabla}_\sigma \overline{\nabla}_\nu (K_\mu{}^\sigma{}_\lambda \xi^\lambda) - \overline{\nabla}_\mu \overline{\nabla}^\sigma (K_{\nu\sigma\lambda} \xi^\lambda) \\ &\quad + \overline{\nabla}_\mu \overline{\nabla}_\nu (K_\alpha{}^\sigma \xi^\alpha) + (2K_{\nu\sigma\rho} K^{\sigma\tau\rho} - K_\sigma K_\nu{}^\tau{}^\sigma) K_{\mu\tau\lambda} \xi^\lambda \\ &\quad - (K_{\mu\nu\rho} K^{\sigma\tau\rho} + K_\nu{}^\sigma{}_\rho K_\mu{}^\tau{}^\rho) K_{\sigma\tau\lambda} \xi^\lambda].\end{aligned}\tag{29}$$

In this manner, Eqs. (25), and (29), convert Eq. (23) into a second-order linear differential equations for the deformation vector $(K_{\mu\nu\alpha})\xi^\alpha$, that we can express in a compact form as

$$(\mathcal{O} \xi^\alpha)_\nu = 0, \quad (30)$$

where the linear operator \mathcal{O} maps vector fields into themselves.

Let us prove now the self-adjointness of the linear operator \mathcal{O} in Eq. (30) [8], and we consider two vector fields ξ^α , and ξ^α , which will be identified finally with solutions of Eqs. (30).

Considering Eq. (25), the contraction of Eq. (23) or (30) with ξ^ν gives a term that does not involve differential operators:

$$(\xi^\nu K_{\sigma\rho\nu})R \bar{\mathcal{C}}^{\mu\nu\sigma\rho} \delta g_{\mu\nu} = -2R \bar{\mathcal{C}}^{(\mu\nu)(\sigma\rho)} (K_{\sigma\rho\beta} \xi^\beta)(K_{\mu\nu\alpha} \xi^\alpha), \quad (31)$$

where we have employed the orthogonal gauge. For the terms involving differential operators we can use differential identities of the form

$$\begin{aligned} K_{\sigma\rho\beta} \xi^\beta \bar{\nabla}_\lambda \bar{\nabla}^\lambda (K_{\mu\nu\alpha} \xi^\alpha) &\equiv \bar{\nabla}_\lambda [K_{\sigma\rho\beta} \xi^\beta \bar{\nabla}^\lambda (K_{\mu\nu\alpha} \xi^\alpha) - \bar{\nabla}^\lambda (K_{\sigma\rho\beta} \xi^\beta) K_{\mu\nu\alpha} \xi^\alpha] \\ &\quad + \bar{\nabla}_\lambda \bar{\nabla}^\lambda (K_{\sigma\rho\beta} \xi^\beta) K_{\mu\nu\alpha} \xi^\alpha, \end{aligned} \quad (32)$$

and hence, considering Eqs. (30) and (31), we can find, after some arrangements, that

$$\xi^\nu (\mathcal{O} \xi^\alpha)_\nu - (\mathcal{O} \xi^\nu)_\alpha \xi^\alpha = \bar{\nabla}_\mu \bar{\mathcal{J}}^\mu, \quad (33)$$

or more explicitly

$$\sigma_1 K_{\sigma\rho\nu} \xi^\nu \bar{\mathcal{C}}^{\mu\nu\sigma\rho} (2\delta R_{\mu\nu} - R\delta g_{\mu\nu}) - \sigma_1 \bar{\mathcal{C}}^{\mu\nu\sigma\rho} (2\delta R'_{\mu\nu} - R\delta g'_{\mu\nu}) K_{\sigma\rho\alpha} \xi^\alpha = \bar{\nabla}_\mu \bar{\mathcal{J}}^\mu, \quad (34)$$

where $\delta R'_{\mu\nu}$ corresponds to $\delta R_{\mu\nu}$ with argument ξ^α , and similarly for $\delta g'_{\mu\nu}$. Moreover,

$$\begin{aligned} \frac{1}{2} \bar{\mathcal{J}}^\mu &= K^{\lambda\tau}{}_\beta \xi^\beta \bar{\nabla}^\mu (K_{\lambda\tau\alpha} \xi^\alpha) - K^{\nu\lambda}{}_\beta \xi^\beta \bar{\nabla}_\lambda (K_{\nu}{}^\mu{}_\alpha \xi^\alpha) \\ &\quad - K^{\mu\nu}{}_\beta \xi^\beta \bar{\nabla}_\lambda (K_{\nu}{}^\lambda{}_\alpha \xi^\alpha) + K^{\mu\nu}{}_\beta \xi^\beta \bar{\nabla}_\nu (K_\alpha{}^\mu \xi^\alpha) \\ &\quad - \bar{\nabla}_\nu (K^{\mu\nu}{}_\beta \xi^\beta) K_\alpha{}^\mu \xi^\alpha + K^{\lambda\nu\mu} K_{\lambda\nu\beta} K_\alpha{}^\mu (\xi^\beta \xi^\alpha) \\ &\quad - K_\beta{}^\mu \xi^\beta \bar{\nabla}^\mu (K_\alpha{}^\mu \xi^\alpha) - (\xi^\alpha \leftrightarrow \xi^\alpha); \end{aligned} \quad (35)$$

in this manner, $\bar{\mathcal{J}}^\mu$ contains all arguments of total divergences such as the first term on the right-hand side of Eq. (32).

Finally, from Eqs. (33), or (34), we conclude that, if $\dot{\xi}^\alpha$ and ξ^α correspond to a pair of solutions admitted by the deformation dynamics (30), $\overline{\mathcal{J}}^\mu$ is worldsheet covariantly conserved

$$\overline{\nabla}_\mu \overline{\mathcal{J}}^\mu = 0. \quad (36)$$

As we shall see, $\overline{\mathcal{J}}^\mu$ will correspond to the integral kernel of a symplectic structure for the topological term, and the property (36) will make sense then. To finish this section, it is worth to point out that $\overline{\mathcal{J}}^\mu$ and its property (36) emerge directly from the nontrivial fluctuations of the topological term, which will constitute an important ingredient in the final setting of our results.

V. The symplectic structure for χ

Once we know the deformations of the internal connection (28), one can find explicitly the deformation of the symplectic potential (18), which constitute the integral kernel of the symplectic structure of the topological term [2]. Considering Eq. (18), we have

$$\delta\psi_{top}^\mu = \delta n^{\alpha\beta} \delta \rho_\alpha{}^\mu{}_\beta - n_\beta^\alpha \delta n^{\mu\tau} \delta \rho_\alpha{}^\beta{}_\tau - n^{\mu\tau} \delta n_\beta^\alpha \delta \rho_\alpha{}^\beta{}_\tau,$$

where the last term vanishes because δn_β^α is proportional to \perp_β^σ (orthogonal to the world-surface), and $\delta \rho_\alpha{}^\beta{}_\tau$ is proportional to n_λ^β (tangential to the world-surface). Using now Eqs. (15), and (28), one obtains that

$$\begin{aligned} \delta(\sqrt{-\gamma} \psi_{top}^\mu) = & 2\sqrt{-\gamma} [K^{\lambda\tau}{}_\beta \xi^\beta \overline{\nabla}^\mu (K_{\lambda\tau\alpha} \xi^\alpha) - K^{\nu\lambda}{}_\beta \xi^\beta \overline{\nabla}_\lambda (K_{\nu}{}^\mu{}_\alpha \xi^\alpha) \\ & - K^{\mu\nu}{}_\beta \xi^\beta \overline{\nabla}_\lambda (K_{\nu}{}^\lambda{}_\alpha \xi^\alpha) + K^{\mu\nu}{}_\beta \xi^\beta \overline{\nabla}_\nu (K_\alpha \xi^\alpha) \\ & - \overline{\nabla}_\nu (K^{\mu\nu}{}_\beta \xi^\beta) K_\alpha \xi^\alpha + K^{\lambda\nu\mu} K_{\lambda\nu\beta} K_\alpha (\xi^\beta \xi^\alpha) \\ & - K_{\beta\xi}{}^\beta \overline{\nabla}^\mu (K_\alpha \xi^\alpha)]; \end{aligned} \quad (37)$$

which is essentially the current found in Eq. (35) if we set up $\dot{\xi}^\alpha = \xi^\alpha$ [8]. Hence, the symplectic structure ω for χ can be written as [2]

$$\omega = \int_\Sigma \delta(\sqrt{-\gamma} \psi_{top}^\mu) d\Sigma_\mu = \int_\Sigma \sqrt{-\gamma} \overline{\mathcal{J}}^\mu d\Sigma_\mu, \quad (38)$$

the first form proves, as we already know [2], that ω is closed, and the second one proves that is independent on the choice of Σ , due to the property (36).

In [1, 2, 8], Σ is defined as “a spacelike section of the worldsheet corresponding to a Cauchy surface for the configuration of the string.” What is Σ for a topological term without dynamics? The nontrivial deformation dynamics comes to rescue: ω is finally defined in terms of field deformations (which, in turn, are defined in terms of the solutions for the deformation dynamics); therefore, Σ will be in this case a spacelike surface for the configuration of the deformation fields appearing in ω .

VI. ω and the symplectic structure of an Abelian gauge theory

In this section we shall show the analogy between the symplectic structure found in the present treatment for the Euler characteristic, and that found in [9] for an Abelian gauge theory within a covariant canonical formalism.

In [9], it is found that

$$\hat{\omega} = \int_{\Sigma} \text{Tr}(\delta F^{\mu\alpha} \delta A_{\mu}) d\Sigma_{\alpha}, \quad (39)$$

is a covariant and gauge invariant symplectic structure for Yang-Mills theory; $F_{\mu\nu}$ is the usual Yang-Mills curvature, A_{μ} the corresponding gauge connection, and Σ a spacelike hypersurface. As we know, F is the *exterior derivative* of A ,

$$F = \mathcal{D}A, \quad (40)$$

and F is also the solution of the equations

$$[\nabla_{\mu}, F^{\mu\nu}] = 0. \quad (41)$$

Eqs. (39), (40), and (41) contain, of course, the Abelian gauge theory as a particular case.

On the other hand, within the differential geometry of the imbedded surface that we are employing, the fact that the embedding has dimension 2 (and that applies for the present case), implies that the inner rotation group is Abelian [5], the analogue of the Abelian (internal) gauge group. Moreover, in this particular case the imbedding two-surface is characterized by the antisymmetric unit tangent element tensor given by [5]

$$\mathcal{E}^{\mu\nu} = \mathcal{E}^{[\mu\nu]} = \mathcal{E}^{AB} i_A^{\mu} i_B^{\nu}, \quad (42)$$

\mathcal{E}^{AB} being the constant components of the standard two-dimensional flat space alternating tensor; $\mathcal{E}^{\mu\nu}$ will be, of course, the analogue of $F^{\mu\nu}$. However, the analogy is not only at the level of the

antisymmetry property of \mathcal{E} and F , since using the tangential derivative of \mathcal{E} given in [5] by

$$\overline{\nabla}_\sigma \mathcal{E}^{\mu\nu} = 2K_{\sigma\tau}^{[\nu} \mathcal{E}^{\mu]\tau}, \quad (43)$$

we can obtain the “wold-surface divergence” of \mathcal{E} contracting Eq. (43) as

$$\overline{\nabla}_\mu \mathcal{E}^{\mu\nu} = K_{\mu\tau}{}^\nu \mathcal{E}^{\mu\tau} - K_{\mu\tau}{}^\mu \rho_\mu{}^\alpha{}_\beta \mathcal{E}^\beta{}_\alpha = 0, \quad (44)$$

the first term vanishes because of the symmetry of K in $(\mu\tau)$ and the antisymmetry of $\mathcal{E}^{\mu\tau}$; the second term because $K_{\mu\tau}{}^\mu = 0$ (see Eq. (11)). Equation (44) is then the analogue of the field equations (41) for the Abelian case.

Continuing, in [5] it is shown that $R\mathcal{E}$ is the *exterior derivative* of the (locally defined, frame gauge dependent) *one-form* $\rho_\mu = \rho_\mu{}^\alpha{}_\beta \mathcal{E}^\beta{}_\alpha$ in the embedding surface,

$$R \mathcal{E} = \overline{\partial} \rho, \quad (45)$$

where $\overline{\partial}$ is the exterior derivative projected on the imbedding surface [5]; in components, Eq. (45) takes the form [5]

$$R \mathcal{E}_{\kappa\lambda} = 2n_{[\lambda}{}^\sigma \overline{\nabla}_{\kappa]} \rho_\sigma, \quad (46)$$

which is the analogue of (40), and ρ_μ is thus the analogue of the gauge connection A_μ .

In order to complete our pretended analogy, it remains to be seen if our ω takes the form (39) given for a gauge theory. This is effectively the case: the topological invariant can be rewritten as [5, 6]

$$\chi = \int R d\overline{\Sigma} = \int \overline{\nabla}_\mu (\mathcal{E}^{\mu\nu} \rho_\nu) d\overline{\Sigma}, \quad (47)$$

and in [2], considering again the gauge nature of ρ , we shown that the variation of χ reads

$$\delta \chi = \int \overline{\nabla}_\mu (\mathcal{E}^{\mu\nu} \delta \rho_\nu) d\overline{\Sigma}, \quad (48)$$

and hence, we identified $(\sqrt{-\gamma})\mathcal{E}^{\mu\nu}\delta\rho_\nu$ as a symplectic potential for χ , which corresponds exactly to ψ_{top}^μ described above [2]. This implies that ω can be written also as

$$\begin{aligned} \omega = \int_\Sigma \delta(\psi_{top}^\mu) d\overline{\Sigma}_\mu &= \int_\Sigma \delta(\sqrt{-\gamma} \mathcal{E}^{\mu\nu} \delta \rho_\nu) d\overline{\Sigma}_\mu \\ &= \int_\Sigma \delta(\sqrt{-\gamma} \mathcal{E}^{\mu\nu}) \delta \rho_\nu d\overline{\Sigma}_\mu, \end{aligned} \quad (49)$$

where we have considered in the last equality the nilpotency of the exterior derivative [2]. Therefore ω takes exactly the required form (39).

Therefore ω mimics entirely $\hat{\omega}$ in the relevant aspects of its mathematical structure.

VII. ω mimics $\hat{\omega}$ in the symmetry properties

Following the idea of the covariant canonical formalism of preserving manifestly the relevant symmetries of the theory, in [9] it is proved that the symplectic form (39) is a covariant and gauge invariant geometrical structure for the gauge theory. Specifically, under the ordinary gauge transformation,

$$A_\mu \rightarrow A_\mu + \partial_\mu \phi, \quad (50)$$

δA_μ and $\delta F^{\mu\nu}$ transform homogeneously, and thus $\hat{\omega}$ in Eq. (39) is gauge invariant. Moreover, the gauge directions on the phase space are defined by the transformation

$$\delta A_\mu \rightarrow \delta A_\mu + \partial_\mu \phi, \quad (51)$$

and $\hat{\omega}$ proves to be also invariant under (51), which defines $\hat{\omega}$ on the reduced phase space $Z \equiv \hat{Z}/G$, where \hat{Z} is the space of solutions of Eqs. (41), and G the group of gauge transformations [9].

We analyze now the analogous situation in our present case for the topological invariant χ . Since the analogy seems to be filled out, we can guess the corresponding gauge transformation for the gauge connection ρ_ν (in an arbitrary curved background),

$$\rho_\nu \rightarrow \rho_\nu + \bar{\nabla}_\nu \phi, \quad (52)$$

where ϕ is an arbitrary scalar field; $\bar{\nabla}_\nu$ is the appropriate derivative since ρ_ν is of support confined to the imbedding surface (see Eq. (3)). Considering the expression (47) for χ , one easily verifies that effectively (52) is a symmetry of the original action:

$$\chi' = \int \bar{\nabla}_\mu [\mathcal{E}^{\mu\nu} (\rho_\nu + \bar{\nabla}_\nu \phi)] d\bar{\Sigma} = \chi + \int \mathcal{E}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \phi d\bar{\Sigma}, \quad (53)$$

where we have considered Eq. (44). Now, taking into account that $\nabla_{[\mu} \nabla_{\nu]} \phi = 0$, we can find easily that

$$\bar{\nabla}_{[\mu} \bar{\nabla}_{\nu]} \phi = K_{[\mu}{}^\sigma{}_{\nu]} \bar{\nabla}_\sigma \phi, \quad (54)$$

and considering that additionally $\mathcal{E}^{\mu\nu} K_{\alpha\beta\nu} = 0$, the integrand in the second term in (53) vanishes, and then $\chi' = \chi$. Note that χ is strictly invariant, it does not change by a total divergence.

Additionally, $R\mathcal{E}_{\mu\nu}$ in Eq. (46) is also invariant, as expected, under the gauge transformation (52),

$$(R\mathcal{E}_{\kappa\lambda})' = R\mathcal{E}_{\kappa\lambda} + 2[\bar{\nabla}_{[\kappa}\bar{\nabla}_{\lambda]}\phi + K_{[\lambda}{}^{\sigma}{}_{\kappa]}\bar{\nabla}_{\sigma}\phi], \quad (55)$$

where the second term vanishes due again to (54).

We show now that ω in (49) retains these symmetries. Since $\delta\rho_\nu$ transforms homogeneously under (52), ω is automatically gauge invariant ($\delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu})$ does not depend on ρ_ν). The situation becomes interesting when we consider the gauge transformation in *field space*,

$$\delta\rho_\nu \rightarrow \delta\rho_\nu + \bar{\nabla}_\nu\phi, \quad (56)$$

the analogue of (51). ω undergoes the transformation

$$\begin{aligned} \omega' &= \int_{\Sigma} \delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu})(\delta\rho_\nu + \bar{\nabla}_\nu\phi) d\bar{\Sigma}_\mu = \omega + \int_{\Sigma} \delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu}) \bar{\nabla}_\nu\phi d\bar{\Sigma}_\mu \\ &= \omega + \int_{\Sigma} \bar{\nabla}_\nu[\phi \delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu})] d\bar{\Sigma}_\mu - \int_{\Sigma} \phi \bar{\nabla}_\nu[\delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu})] d\bar{\Sigma}_\mu, \end{aligned} \quad (57)$$

therefore, this equation implies that ω will change by a total divergence, the second term on the right hand-side, if the last one vanishes. Since ϕ is an arbitrary field, the only possibility is that $\bar{\nabla}_\nu(\delta(\sqrt{-\gamma}\mathcal{E}^{\mu\nu}))$ vanishes. Let us prove that this is effectively the situation.

We take first the variation of Eq. (44):

$$\delta(\bar{\nabla}_\mu\mathcal{E}^{\mu\nu}) = \bar{\nabla}_\mu \delta\mathcal{E}^{\mu\nu} + \mathcal{E}^{\lambda\nu} n_\mu^\alpha \delta\Gamma_{\alpha\lambda}^\mu + \mathcal{E}^{\lambda\nu} K_\lambda{}^{\rho\sigma} \delta g_{\rho\sigma} = 0, \quad (58)$$

where we have considered Eq. (43), the antisymmetry of $\mathcal{E}^{\mu\nu}$, and the symmetry of $\delta\Gamma_{\alpha\lambda}^\nu$ in $(\alpha\lambda)$.

On the other hand, considering that

$$\delta\Gamma_{\alpha\lambda}^\mu = \frac{1}{2}g^{\mu\rho}(\nabla_\alpha\delta g_{\lambda\rho} + \nabla_\lambda\delta g_{\alpha\rho} - \nabla_\rho\delta g_{\alpha\lambda}), \quad (59)$$

one can show that

$$n_\lambda^\beta n_\mu^\alpha \delta\Gamma_{\alpha\beta}^\mu = \frac{1}{2}n_\lambda^\beta n^{\alpha\rho} \nabla_\beta \delta g_{\alpha\rho} = \bar{\nabla}_\lambda \left(\frac{\delta\sqrt{-\gamma}}{\sqrt{-\gamma}} \right) - K_\lambda{}^{\sigma\rho} \delta g_{\sigma\rho}, \quad (60)$$

where we take into account Eq. (15). Thus, from Eqs. (58), and (60), we have

$$\bar{\nabla}_\mu(\delta\mathcal{E}^{\mu\nu}) = -\frac{\mathcal{E}^{\mu\nu}}{\sqrt{-\gamma}}\bar{\nabla}_\mu(\delta\sqrt{-\gamma}), \quad (61)$$

which implies finally that

$$\bar{\nabla}_\mu[\delta(\sqrt{-\gamma})\mathcal{E}^{\mu\nu}] = 0, \quad (62)$$

as required for the vanishing of the last term on the right hand-side in Eq. (57); the second term in Eq. (57) reduces to

$$\int_\Sigma \bar{\nabla}_\nu[\phi\delta(\sqrt{-\gamma})\mathcal{E}^{\mu\nu}]d\bar{\Sigma}_\mu = \int_{\partial\Sigma} \phi\delta(\sqrt{-\gamma})\mathcal{E}^{\mu\nu}d\bar{\Sigma}_{\mu\nu}, \quad (63)$$

which vanishes if we assume that the field variations are of compact support at the boundary $\partial\Sigma$ (analogous boundary conditions are imposed on the field variations in the case of gauge theory [9]). Under these conditions, ω has not components along the gauge orbits, and then we have defined ω on certain *reduced phase space*. Note that, up to here, we have not defined strictly the phase space, because we have not equations of motion for χ ! On what space the pre-symplectic form (49), and the subsequent symplectic structure are defined then?

VIII. Phase space and reduced phase space for a topological invariant

In the original work on a covariant description of the canonical formalism [9], the phase space is defined as *the space of solutions of the equations of motion*, and the reduced or physical phase space as the phase space modulo gauge transformations. However, in the present case, one has only a topological term without dynamics, and we can not apply such a definition.

In order to find a covariant definition of phase space for this case, we can use some ideas known in the literature, and the analogy here established. We can define first the *kinematic phase space* Z as the space of all smooth ρ connections and \mathcal{E} fields, on which the pre-symplectic form (49) will be defined. If the equations (44), and (46) for \mathcal{E} and ρ are satisfied, we obtain a sub-manifold of Z that we can call phase space \bar{Z} . The reduced phase space \hat{Z} is obtained then from \bar{Z} by dividing it by the volume of the symmetry group, we mean \hat{Z} is constituted by the set of equivalence classes, where the equivalence relation is given by the gauge transformations (56): two points on the phase

space \overline{Z} are equivalent if they differ by a gauge transformation. In conclusion, in a covariant canonical formalism for the topological invariant χ , the equations (44), and (46), which characterize the two-dimensional embedding surface, play the role, in some sense, of the nonexistent equations of motion. The analogy is in this sense also filled out, since the analogue of Eqs. (44), and (46) for χ correspond to the equations of motion (41) and the Bianchi identity (40) for gauge theory.

IX. Remarks and prospects

We remark first that the results established in this work are valid for a two-dimensional surface embedded in a curved background of arbitrary dimension. Therefore, considering that the symplectic structure constructed for χ will govern finally the transition between the classical and quantum domains, it is inevitable to wonder about questions such as the critical dimension of the background, preservation of the classical symmetries throughout the quantization process, etc, which turn out to be of particular interest in string theory. For example, a way in which the phenomenon of the critical dimension is manifested in (bosonic) string theory is that the corresponding Poincaré algebra at a quantum level is closed only if the background dimension is 26. One can ask if χ has a relevant effect on this particular question. In this sense, one can study the unexpected nontrivial realizations of the Poincaré algebra (at classical level) for χ from its symplectic structure ω , as a first step in such a direction. In fact, this will be the subject of forthcoming works.

Considering the point of view, initiated by Dirac, that the classical \leftrightarrow quantum correspondence of the physical systems should be formulated in terms of analogies between their mathematical structures, the similarly established here with an Abelian gauge theory (perhaps the most studied gauge theory), allows to reveal the role of the topological terms in a quantum domain. Conversely, the possible analogy with a more general non-Abelian gauge theory may be useful in order to construct topological invariants in higher dimensional systems.

As discussed in [1, 2], there exists another topological invariant associated with a two-dimensional surface (embedded in a four-dimensional background), the so called first Chern number of the normal bundle of the surface, and related geometrically with the number of self-intersections of the surface. The corresponding symplectic structure proves to have also the form of an Abelian gauge theory, in an entirely similar way to the case developed here; however, explicit calculations will be performed

in its opportunity.

Additionally we comment that in Ref. [10], it is proved the conformal invariance of the phase space formulation presented in this work.

Finally, the results obtained here, can be also achieved using the symplectic scheme developed by Carter in [7].

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